Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

That primes are quite irregularly distributed, will be plain from this:

The 26,379th hundred contains no prime.
The 27,050th hundred contains 17 primes.

Even when they are taken in large masses, the irregularity is still apparent, thus:—

The 27th hundred thousand contains 6,762 primes.
The 28th hundred thousand contains 6,714 primes.
The 29th hundred thousand contains 6,744 primes.

For the numerical data here given I am indebted to Gauss, (Vol. above quoted.) G. W. Hill.

* * *

The formula $N = \frac{x}{A \log. x - B}$,

referred to by Prof. Brooks in No. 3, requires that the logarithm should be hyperbolic, which he neglects to state.

This formula is fully and ably discussed in Vol. II, § VIII, p. 65 of *Legendre's Theorie des Nombres*, 3d Edition, 1830.

The formula adopted by him is

$$N = \frac{x}{h. \log. x - 1.08366},$$

Many years since I became interested in this matter, and I then used the formula

$$N = \frac{x}{1.001 h. \log. x - 1.09}.$$

G. R. P.

* * *

SOLUTIONS OF PROBLEMS IN NO. 2.

Solutions have been received as follows: R. J. Adcock solved 7; Irving P. Church solved 6; G. W. Hill solved 9; Prof. A. Hall solved 10; H. Heaton solved 6, 7, 8, 9 and 10; Prof. D. Kirkwood solved 7; Artemas Martin solved 5, 6 and 7; Prof. J. Scheffer solved 5 and 7; James Stott solved 6; Walter Siverly solved 8; and Prof. W. Wylie solved 7. David Wickersham and William Hoover sent solutions to Problems in No. 1 too late for notice in No. 3.

5. "In a plane triangle there are given the three lines bisecting the angles, a, b and c, to find the sides."

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

Let $CF = a$, $BE = b$, $AD = c$ be the three bisecting lines, and let

x, y and z represent the three corresponding sides. According to a well known theorem we have $y : z :: A\,F : B\,F :: A\,F : x - A\,F$, whence

$$A\,F = \frac{x\,y}{y+z} \text{ and } B\,F = \frac{x\,z}{y+z}.$$

It is further $\cos \tfrac{1}{2} C = \dfrac{y^2 + a^2 - \overline{AF}^2}{2\,a\,y} = \dfrac{z^2 + a^2 - \overline{BF}^2}{2\,a\,z}$.

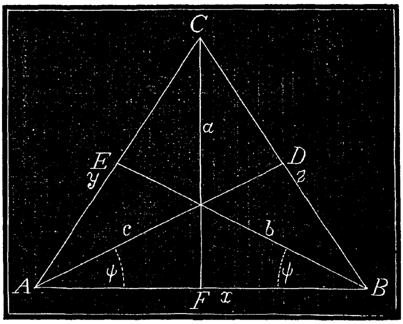
Substituting the values for $A\,F$ and $B\,F$, we obtain

$$a^2 = y\,z\,\frac{(x+y+z)\,(y+z-x)}{(y+z)^2},$$

and in the same way we get

$$b^2 = x\,z\,\frac{(x+y+z)\,(x+z-y)}{(x+z)^2}, \quad c^2 = x\,y\,\frac{(x+y+z)\,(x+y-z)}{(x+y)^2}.$$

These three elegant expressions can be very usefully employed for the computation of the three bisecting lines, if the sides are given; the reverse operation, however, leads to an equation in x, or y, or z, the degree of which renders the solution, in a general form, a matter of impossibility. Even if a, b and c have numerical values, the solution would prove very fatiguing for the most patient of all reckoners. We may, therefore, consider the solution for three different bisecting lines impracticable, and we propose to give a full solution for the case of an isosceles triangle.



Let $AB\,C$ represent an isosceles triangle ($A\,C = B\,C$) and $B\,E = A\,D = b$, $C\,F = a$, the bisecting lines. Denoting the angle $E\,B\,A = D\,A\,B$ by ψ, we have in the triangle $A\,B\,E$,

$$b : A\,B :: \sin 2\,\psi : \sin 3\,\psi,$$

but $A\,B = 2\,A\,F = 2\,a \operatorname{cotang} 2\,\psi$. hence we have the equation

$$2\,a \cos 2\,\psi = b \sin 3\,\psi.$$

After an easy transformation we get

$$\sin^3 \psi - \frac{a}{b} \sin^2 \psi - \tfrac{3}{4} \sin \psi + \frac{a}{2b} = 0.$$

Putting $\sin \psi = x + \dfrac{a}{3b}$, we obtain

$$x^3 - \frac{4\,a^2 + 9\,b^2}{12\,b^2}\,x - \frac{8\,a^3 - 27\,a\,b^2}{108\,b^3} = 0.$$

Of this equation I wish to give a short solution: Putting for brevity's sake

$$\frac{4\,a^2+9\,b^2}{12\,b^2}=p \text{ and } \frac{8\,a^3-27\,a\,b^2}{108\,b^3}=q$$

we get $x^3 - p\,x - q = 0$. Placing

$$\frac{(8\,a^3-27\,a\,b^2)^2}{(4\,a^2+9\,b^2)^3}=\frac{27\,q^2}{4\,p^3}=\cos^2\lambda;$$

and taking into consideration that $4\,p^3 > 27\,q^2$, we obtain according to Cardan's formula, after some easy operations,

$$x=\sqrt{\frac{p}{3}}\left(\sqrt[3]{\cos\lambda+\sin\lambda\,.\,\sqrt{-1}}+\sqrt[3]{\cos\lambda-\sin\lambda\,.\,\sqrt{-1}}\right)$$

$$=\sqrt{\frac{p}{3}}\left((\cos\tfrac{1}{3}\lambda+\sin\tfrac{1}{3}\lambda\,.\,\sqrt{-1})+(\cos\tfrac{1}{3}\lambda-\sin\tfrac{1}{3}\lambda\,\sqrt{-1})\right)$$

$$=2\sqrt{\frac{p}{3}}\cos\tfrac{1}{3}\,\lambda.$$

But it is $\cos\lambda = \cos(360^\circ+\lambda) = \cos(720^\circ+\lambda)$, hence we obtain the three roots,

$$x_1=2\sqrt{\frac{p}{3}}\cos\tfrac{1}{3}\lambda,\qquad x_2=2\sqrt{\frac{p}{3}}\cos\left(120^\circ+\frac{\lambda}{3}\right),$$

$$x_3=2\sqrt{\frac{p}{3}}\cos\left(240^\circ+\frac{\lambda}{3}\right).$$

There are some pretty geometrical demonstrations of the theorem, that the triangle is an isosceles one if two of its bisecting lines are equal.

[In solving this question as proposed, Mr. Martin found three equations representing the values of a^2, b^2 and c^2 in functions of x, y and z, which are identical with the foregoing equations determined by Prof. Scheffer.]

6. "Find a convenient formula for calculating the capacity of a cistern constructed as follows, viz: Having a lower concavity which is a shperical segment whose versed sine is a and chord $2\,r$, a central cylindrical part whose radius is r and perpendicular height h, and an upper concavity which is a spherical segment whose versed sine is b and chord $2\,r$."

SOLUTION BY IRVING P. CHURCH, NEWBURGH, N. Y.

$$V=\frac{\pi}{6}[(3a+3b+6h)r^2+(a^3+b^3)]=.523598[(3a+3b+6h)r^2+(a^3+b^3)].$$

7. "Multiply $\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\&c.}}}$ by $\sqrt{1+\cfrac{1}{\sqrt{1+\cfrac{1}{\sqrt{1+\&c.}}}}}$

and express the product in a finite number of terms."

SOLUTION BY PROF. D. KIRKWOOD.

Putting the first factor equal to x, we have

$$x = \sqrt[3]{1+x}, \text{ or, } x^3 - x = 1. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{Assuming } x = p^{\frac{1}{3}} + \frac{1}{3p^{\frac{1}{3}}},$$

$$p^2 - p = -\tfrac{1}{27}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\text{therefore, } p = \tfrac{1}{18}\,(9 \pm \sqrt{69}), \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{and } x = [\tfrac{1}{18}(9 \pm \sqrt{69})]^{\frac{1}{3}} + \frac{1}{3[\frac{1}{18}(9 \pm \sqrt{69})]^{\frac{1}{3}}}.$$

Again: Putting the second factor $=$ to x', we have

$$x' = \sqrt{1 + \frac{1}{x'}}; \text{ or } x'^3 - x' = 1. \quad\ldots\ldots\ldots\ldots\ldots(4)$$

Consequently $x = x'$, and the product of the factors is

$$\left\{ [\tfrac{1}{18}(9 \pm \sqrt{69})]^{\frac{1}{3}} + \frac{1}{3[\frac{1}{18}(9 \pm \sqrt{69})]^{\frac{1}{3}}} \right\}^2.$$

[All the solutions to this question were very nearly alike. After finding the equation $x^3 - x = 1$, Prof. Scheffer remarks, "we might now find the only real root of this equation and then square it; but by substituting $\sqrt{y}$ in place of x we find the square of x at once. In this way we obtain the equation $y^3 - 2\,y^2 + y - 1 = 0$; from which we find $y = 1.75488$."]

---

8. "A ball *rolls* down the convex surface of a fixed sphere, the friction being just sufficient to prevent sliding; find the point where it leaves the sphere."

SOLUTION BY H. HEATON.

Assume radius of the sphere $= 1$, and let $h =$ vertical distance the ball rolls before leaving the sphere. Then the velocity of the ball

$$= \sqrt{\frac{10}{7}\,gh},$$

and its horizontal velocity

$$= (1-h)\sqrt{\frac{10}{7}\,gh}.$$

Hence the equation to the parabola in which the ball moves when it leaves the sphere is

$$y^2 = \frac{20}{7}\,(1-h)^2 x.$$

But when it leaves the sphere the radius of curvature of the parabola equals the radius of the sphere.

$$\therefore \frac{10}{7} \cdot \frac{h}{1-h} = 1; \quad \therefore h = \frac{7}{17}$$

9. "P and Q denoting two entire functions of x, such that we have $\sqrt{1-P^2} = Q\sqrt{1-x^2}$, we have necessarily

$$\frac{d\,P}{\sqrt{1-P^2}} = n\,\frac{d\,x}{\sqrt{1-x^2}};$$

n denoting an entire number."

SOLUTION BY G. W. HILL.

Let us suppose that the term of P involving the highest power of x is $A\,x^n$, then the term of Q involving the highest power of x must be $A\,x^{n-1}$. It is evident the equatioı$PP = 0$ can have no root common with the roots of $Q = 0$, else would the right hand member of the first given equation vanish without the left hand member doing so. Squaring now and differentiating this equation, we get

$$P\frac{d\,P}{d\,x} = Q\left\{x\,Q - (1-x^2)\frac{d\,Q}{d\,x}\right\}.$$

Since $P = 0$ and $Q = 0$ have no common roots, it is plain that

$$\frac{d\,P}{d\,x} = 0 \text{ and } Q = 0$$

must have *all* their roots common, and therefore, having regard to the fact that the first term of P is $A\,x^n$ and the first of Q, $A\,x^{n-1}$,

$$\frac{d\,P}{d\,x} = n\,Q.$$

If we divide this by the first equation of the proposition we shall have the second.

SOLUTION BY H. HEATON.

From the equation $\sqrt{1-P^2} = Q\sqrt{1-x^2}$, ...(1)

we have $P^2 = Q^2\,x^2 - Q^2 + 1$. ....(2)

Whence

$$\frac{d\,P}{d\,x} = \left[\frac{x\,Q + (x^2-1)\dfrac{d\,Q}{d\,x}}{P}\right]Q.$$

Because the first member of this equation is an entire quantity, it follows that the second member must also be; and as it is evident from equation (2) that P and Q are prime to each other, and

$$\frac{d\,P}{d\,x} \text{ and } Q$$

are of the same degree, it follows that

$$\frac{x\,Q + (x^2-1)\dfrac{d\,Q}{d\,x}}{P}$$

is an entire constant quantity. Hence

$$\frac{d\,P}{d\,x} = n\;Q;$$

$\therefore d\,P = n\;Q\;d\,x$. Dividing this equation by (1) member by member we have

$$\frac{d\,P}{\sqrt{1 - P^2}} = n\,\frac{d\,x}{\sqrt{1 - x^2}}.$$

---

10. "The Chief Justice of a court makes a large number of legal decisions. Afterward it is found that 50 per cent. of these decisions are erroneous. Required to determine the legal knowledge of the Judge."

SOLUTION BY PROF. A. HALL.

If we assume the Judge to be an honest man, and divide the decisions into two classes, viz: right and wrong, it follows from the conditions given and the calculus of probabilities that such a Judge is a perfect legal ignoramus. This result is simple enough, and yet decisions and predictions of this kind not infrequently form the basis of extended reputations, ex. qr in the case of weather prophets, fortune tellers, &c. We forget that one ought to guess right half the time. Some of this delusion may be seen also in the cherishing of astronomical fancies and analogies, and the straining of facts to conform them to these analogies.

* * *

PROBLEMS.

16. What are the sides of a right angled triangle the area of which is 5 acres, and one of its acute angles 22° 30′?—Communicated by Joseph B. Mott, Neosho, Mo.

17. Find the sum of the series

$$a\,r^n + (a + b)\,r^{n-1} + (a + 2\,b)\,r^{n-2} + \ldots (a + n\,b) = S.$$

—Communicated by James Stott, Esq., Carbondale, Pa.

18. Find the differential coefficient of the function $(1+x^m)^n \,.\, (1+x^n)^m$ and express the answer in its simplest form.—Communicated by Philip Hoagland, Newcomerstown, Ohio.

19. Referring to Question 4, (No. 1): At what time will the lower cask contain the greatest quantity of wine?—Communicated by Prof. Geo. R. Perkins.